Chapter 1: Rotational Dynamics

EXERCISES [PAGES 23 - 25]

Exercises | Q 1.1 | Page 23

Choose the correct option.

When seen from below, the blades of a ceiling fan are seen to be revolving anticlockwise and their speed is decreasing. Select the correct statement about the directions of its angular velocity and angular acceleration.

1. **Angular velocity upwards, angular acceleration downwards.**
2. Angular velocity downwards, angular acceleration upwards.
3. Both, angular velocity and angular acceleration, upwards.
4. Both, angular velocity and angular acceleration, downwards.

SOLUTION

Angular velocity upwards, angular acceleration downwards.

Exercises | Q 1.2 | Page 23

Choose the correct option.

A particle of mass 1 kg, tied to a 1.2 m long string is whirled to perform vertical circular motion, under gravity. Minimum speed of a particle is 5 m/s. Consider following statements.

P) Maximum speed must be 55 m/s.

Q) Difference between maximum and minimum tensions along the string is 60 N.

Select the correct option.

1. Only the statement P is correct.
2. Only the statement Q is correct.
3. **Both the statements are correct.**
4. Both the statements are incorrect.

SOLUTION

Both the statements are correct.

Exercises | Q 1.3 | Page 23

Choose the correct option.

Select correct statement about the formula (expression) of moment of inertia (M.I.) in terms of mass M of the object and some of its distance parameter/s, such as R, L, etc.

1. Different objects must have different expressions for their M.I.
2. **When rotating about their central axis, a hollow right circular cone and a disc have the same expression for the M.I.**
3. Expression for the M.I. for a parallelepiped rotating about the transverse axis passing through its centre includes its depth.
4. Expression for M.I. of a rod and that of a plane sheet is the same about a transverse axis.

SOLUTION

When rotating about their central axis, a hollow right circular cone and a disc have the same expression for the M.I.

Exercises | Q 1.4 | Page 23

Choose the correct option.

In a certain unit, the radius of gyration of a uniform disc about its central and transverse axis is $\sqrt{2.5}$. Its radius of gyration about a tangent in its plane (in the same unit) must be

$\sqrt{5}$

2.5

$2\sqrt{2.5}$

$\sqrt{12.5}$

SOLUTION

2.5

Exercises | Q 1.5 | Page 23

Choose the correct option.

Consider the following cases:
(P) A planet revolving in an elliptical orbit.
(Q) A planet revolving in a circular orbit.
Principle of conservation of angular momentum comes in force in which of these?

1. Only for (P)
2. Only for (Q)
3. **For both, (P) and (Q)**
4. Neither for (P), nor for (Q)

SOLUTION

For both, (P) and (Q)

Exercises | Q 1.6 | Page 24

Choose the correct option.

A thin walled hollow cylinder is rolling down an incline, without slipping. At any instant, without slipping. At any instant, the ratio "Rotational K.E.: Translational K.E.: Total K.E." is

1. 1: 1: 2
2. 1: 2: 3
3. 1: 1: 1
4. **2: 1: 3**

SOLUTION

2: 1: 3

Exercises | Q 2.1 | Page 24
Answer in brief:
Why are curved roads banked?

SOLUTION

A car while taking a turn performs circular motion. If the road is level (or horizontal road), the necessary centripetal force is the force of static friction between the car tyres and the road surface.

The friction depends upon the nature of the surfaces in contact and the presence of oil and water on the road. If the friction is inadequate, a speeding car may skid off the road. Since the friction changes with circumstances, it cannot be relied upon to provide the necessary centripetal force. Moreover, friction results in fast wear and tear of the tyres. To avoid the risk of skidding as well as to reduce the wear and tear of the car tyres, the road surface at a bend is tilted inward, i.e., the outer side of the road is raised above its inner side.

This is called banking of road. On a banked road, the resultant of the normal reaction and the gravitational force can act as the necessary centripetal force. Thus, every car can be safely driven on such a banked curve at certain optimum speed, without depending on friction. Hence, a road should be properly banked at a bend.
The angle of banking is the angle of inclination of the surface of a banked road at a bend with the horizontal.

Exercises | Q 2.2 | Page 24
Answer in brief:
Do we need a banked road for a two-wheeler? Explain.

SOLUTION

When a two-wheeler takes a turn along an unbanked road, the force of friction provides the centripetal force. The two-wheeler leans inward to counteract a torque that tends to topple it outward. Firstly, friction cannot be relied upon to provide the necessary centripetal force on all road conditions. Secondly, the friction results in the wear and tear of the tyres. On a banked road at a turn, any vehicle can negotiate the turn without depending on friction and without straining the tyres.

Exercises | Q 2.3 | Page 24
Answer in brief:
On what factors does the frequency of a conical pendulum depend? Is it independent of some factors?

SOLUTION

The frequency of a conical pendulum, of string length L and semi-vertical angle θ is

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{L\cos\theta}}$$

where g is the acceleration due to gravity at the place.

From the above expression, we can see that

(i) $n \propto \sqrt{g}$

(ii) $n \propto \frac{1}{\sqrt{L}}$

(iii) $n \propto \frac{1}{\sqrt{\cos\theta}}$

(if θ increases, cos θ decreases and n increases)

(iv) The frequency is independent of the mass of the bob.

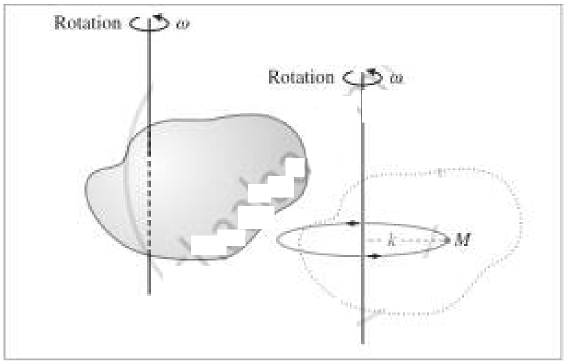
Exercises | Q 2.4 | Page 24

Answer in brief:

Why is it useful to define the radius of gyration?

SOLUTION

Definition: The radius of gyration of a body rotating about an axis is defined as the distance between the axis of rotation and the point at which the entire mass of the body can be supposed to be concentrated so as to give the same moment of inertia as that of the body about the given axis.



The moment of inertia (MI) of a body about a given rotation axis depends upon (i) the mass of the body and (ii) the distribution of mass about the axis of rotation. These two factors can be separated by expressing the MI as the product of the mass (M) and the square of a particular distance (k) from the axis of rotation. This distance is called the radius of gyration and is defined as given above. Thus,

$$I = \sum_i m_i r_i^2 = Mk^2$$

$$\therefore k = \sqrt{\frac{I}{M}}$$

Physical significance: The radius of gyration is less if I is less, i.e., if the mass is distributed close to the axis; and it is more if I is more, i.e., if the mass is distributed away from the axis. Thus, it gives an idea about the distribution of mass about the axis of rotation.

Exercises | Q 2.5 | Page 24

Answer in brief:

A uniform disc and a hollow right circular cone have the same formula for their M.I. when rotating about their central axes. Why is it so?

SOLUTION

A uniform disc and a hollow right circular cone have the same formula for their moment of inertia.

$MI = mr^2/2$

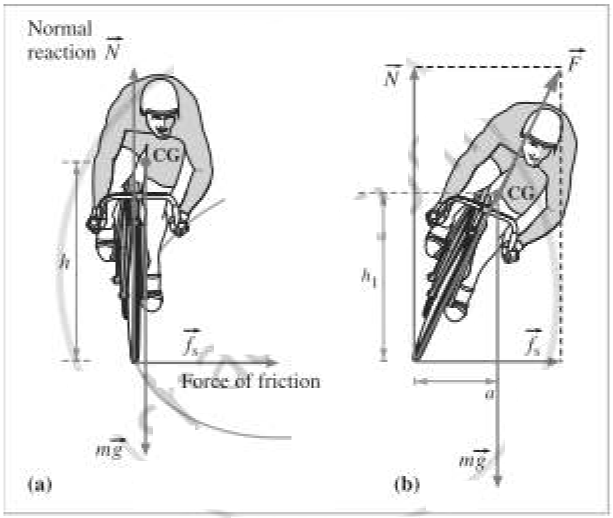
This is because when a hollow right circular cone is cut along its slanting side and the metal is stretched out, you will find that the surface of the cone will form a circle. This shape is the same as the shape of a disc, which is also a circle. Hence they both have the same formula for MI.

Exercises | Q 3 | Page 24

While driving along an unbanked circular road, a two-wheeler rider has to lean with the vertical. Why is it so? With what angle the rider has to lean? Derive the relevant expression. Why such a leaning is not necessary for a four wheeler?

SOLUTION

(i) When a bicyclist takes a turn along an unbanked road, the force of friction $\vec{f}_s$ provides the centripetal force; the normal reaction of the road $\vec{N}$ is vertically up. If the bicyclist does not lean inward, there will be an unbalanced outward torque about the centre of gravity, f s ·h, due to the friction force that will topple the bicyclist outward. The bicyclist must lean inward to counteract this torque (and not to generate a centripetal force) such that the opposite inward torque of the couple formed by $\vec{N}$ and the weight $\vec{g}$, $mg.a = f_s.h_1$



A bicyclist taking a turn to his left on a level road

(ii) Since the force of friction provides the centripetal force,

$$f_s = \frac{mv^2}{r}$$

If the cyclist leans from the vertical by an angle θ, the angle between $\vec{N}$ and $\vec{F}$

$$\tan\theta = \frac{f_s}{N} = \frac{mv^2/r}{mg} = \frac{v^2}{gr}$$

Hence, the cyclist must lean by an angle

$$\theta = \tan^{-1} = \left(\frac{v^2}{gr}\right)$$

(iii) When a car takes a turn along a level road, apart from the risk of skidding off outward, it also has a tendency to roll outward due to an outward torque about the Centre of gravity due to the friction force. But a car is an extended object with four wheels.

So, when the inner wheels just get lifted above the ground, it can be counterbalanced by a restoring torque of the couple formed by the normal reaction (on the outer wheels) and the weight.

Consider a car of mass m taking a turn of radius r along a level road. As seen from an inertial frame of reference, the forces acting on the car are:

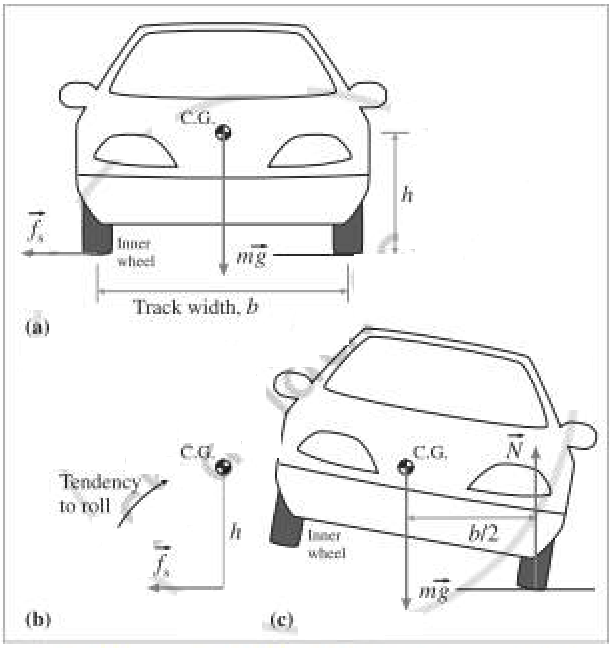
(1) the lateral limiting force of static friction $\vec{f}_s$ on the wheels – acting along the axis of the wheels and towards the center of the circular path which provides the necessary centripetal force.

(2) the weight $\vec{mg}$ acting vertically downwards at the centre of gravity (C.G.)

(3) the normal reaction $\vec{N}$ of the road on the wheels, upwards effectively at the C.G. Since maximum centripetal force = limiting force of static friction,

$$ma_r = \frac{mv^2}{r} = f_s \quad ...(1)$$

In a simplified rigid-body vehicle model, we consider only two parameters – the height h of the C.G. above the ground and the average distance b between the left and right wheels called the track width.



Rolling tendency of a vehicle negotiating a bend on a level road

The friction force $\vec{f}_s$ on the wheels produces a torque τ_t that tends to overturn/rollover the car about the outer wheel in the above figure (b). Rotation about the front-to-back axis is called roll.

$$\tau_t = f_s . h = \left(\frac{mv^2}{r}\right)h \quad ...(2)$$

When the inner wheel just gets lifted above the ground, the normal reaction $\vec{N}$ of the road acts on the outer wheels but the weight continues to act at the C.G. Then, the couple formed by the normal reaction and the weight produces a opposite torque τ_r which tends to restore the car back on all four wheels in the above figure (b).

$$\tau_r = mg. \frac{b}{2} \quad ....(3)$$

The car does not topple as long as the restoring torque τ_r counterbalances the toppling torque τ_r.

Thus, to avoid the risk of rollover, the maximum speed that the car can have is given by

$$\left(\frac{mv^2}{r}\right)h = mg.\frac{b}{2}$$

$$\therefore v_{max} = \sqrt{\frac{rbg}{2h}} \quad ...(4)$$

Thus, vehicle tends to roll when the radial acceleration reaches a point where inner wheels of the four-wheeler are lifted off of the ground and the vehicle is rotated outward.

A rollover occurs when the gravitational force $\overrightarrow{mg}$ passes through the pivot point of the outer wheels, i.e., the C.G. is above the line of contact of the outer wheels. Equation (3) shows that this maximum speed is high for a car with larger track width and lower center of gravity.

Exercises | Q 4.1 | Page 24

Using energy conservation, derive the expressions for the minimum speeds at different locations along a vertical circular motion controlled by gravity?

SOLUTION

Consider a particle of mass m attached to a string and revolved in a vertical circle of radius r. At every instant of its motion, the particle is acted upon by its weight $\overrightarrow{mg}$ and the tension $\overrightarrow{T}$ in the string. The particle may not complete the circle if the string slackens before the particle reaches the top. This requires that the particle must have some minimum speed.

At the top (point A): Let v_1 be the speed of the particle and T_1 the tension in the string. Here, both $\overrightarrow{T}_1$ and weight $\overrightarrow{mg}$ is vertically downward. Hence, the net force on the particle towards the centre O is T_1 + mg, which is the necessary centripetal force.

$$\therefore T_1 + mg = \frac{mv_1^2}{r} \quad ...(1)$$

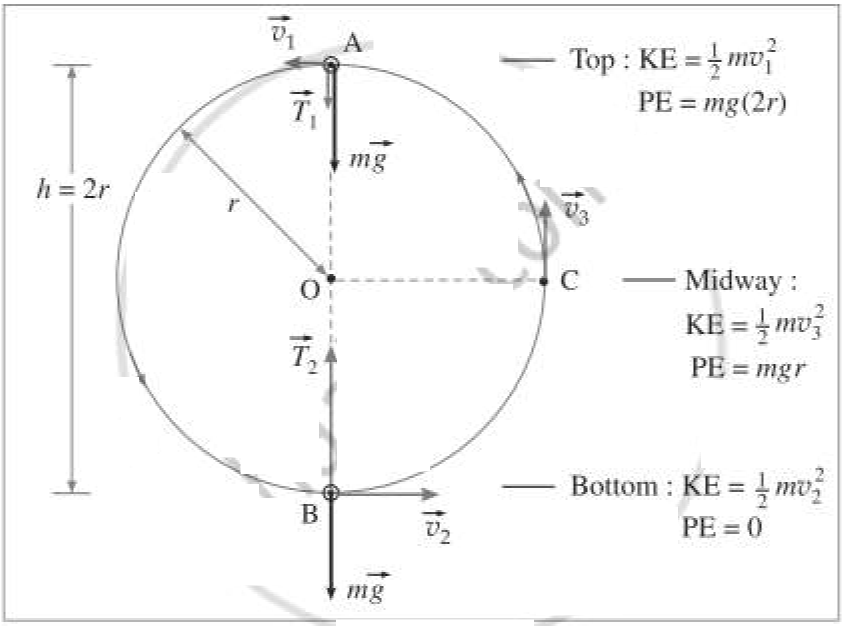
To find the minimum value of v_1 that the particle must have at the top, we consider the limiting case when the tension T_1 just becomes zero.

$$\therefore \frac{mv_1^2}{r} = mg$$

that is, the particle's weight alone is the necessary centripetal force at point A.

$$\therefore v_1^2 = gr$$

$$\therefore v_1 = \sqrt{gr} \quad ...(2)$$



Vertical circular motion (schematic)

At the bottom (point B): Let v_2 be the speed at the bottom. Taking the reference level for zero potential energy to be the bottom of the circle, the particle has only kinetic energy $\frac{1}{2}mv_2^2$ at the lowest point.

Total energy at the bottom = KE + PE

$$= \frac{1}{2}mv_2^2 + 0 = \frac{1}{2}mv_2^2 \quad ....(3)$$

As the particle goes from the bottom to the top of the circle, it rises through a height h = 2r. Therefore, its potential energy at the top is

mgh = mg(2r)

and, from Eq. (2), its minimum kinetic energy there is

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mgr \quad ....(4)$$

Minimum total energy at the top = KE + PE

$$= \frac{1}{2}mgr + 2mgr$$

$$= \frac{5}{2}mgr \quad ....(5)$$

Assuming that the total energy of the particle is conserved, total energy at the bottom = total energy at the top. Then, from Eqs. (4) and (5),

$$\frac{1}{2}mv_2^2 = \frac{5}{2}mgr$$

The minimum speed the particle must have at the lowest position is

$$v_2 = \sqrt{5gr} \quad ...(6)$$

At the midway (point C): Let v be the speed at point C, so that its kinetic energy is $\frac{1}{2}mv_3^2$. At C, the particle is at a height r from the bottom of the circle. Therefore, its potential energy at C is mgr. Total energy at C

$$= \frac{1}{2}mv_3^2 + mgr \quad ...(7)$$

From the law of conservation of energy, total energy at C = total energy at B

$$\therefore \frac{1}{2}mv_3^2 + mgr = \frac{5}{2}mgr$$

$$\therefore v_3^2 = 5gr - 2gr = 3gr$$

∴ The minimum speed the particle must have midway up is

$$v_3 = \sqrt{3gr} \quad ...(8)$$

Exercises | Q 4.2 | Page 24

Is a zero speed possible at the uppermost point? Under what condition/s?

SOLUTION

In a non-uniform vertical circular motion, e.g., those of a small body attached to a string or the loop-the-loop maneuvers of an aircraft or motorcycle or skateboard, the body must have some minimum speed to reach the top and complete the circle.

In this case, the motion is controlled only by gravity and zero speed at the top is not possible. However, in a controlled vertical circular motion, e.g., those of a small body attached to a rod or the giant wheel (Ferris wheel) ride, the body or the passenger seat can have zero speed at the top, i.e., the motion can be brought to a stop.

Exercises | Q 4.3 | Page 24

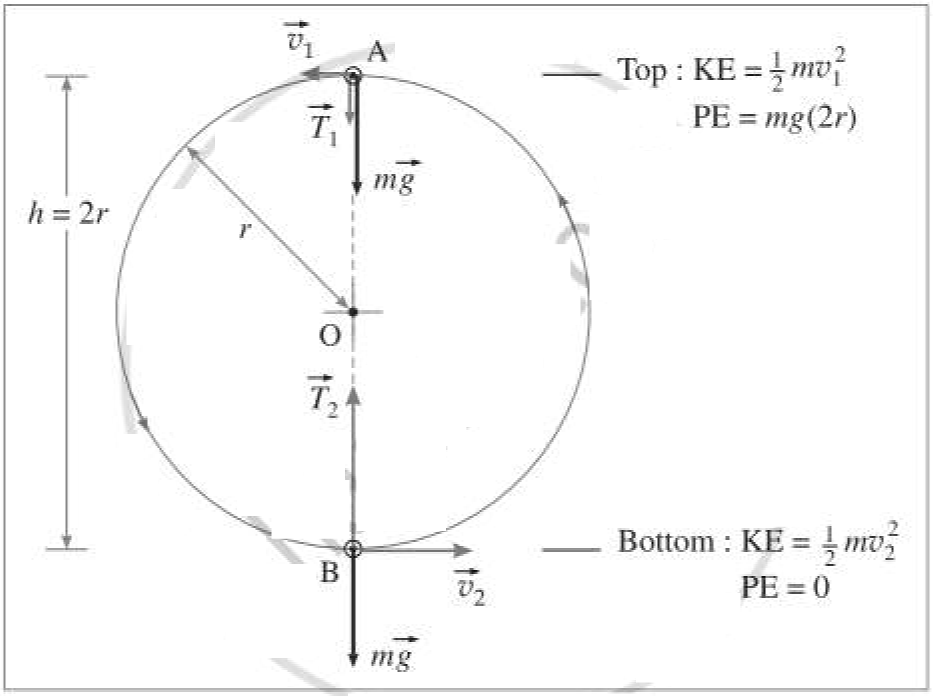
Also prove that the difference between the extreme tensions (or normal forces) depends only upon the weight of the objects.

SOLUTION

Consider a small body (or particle) of mass m tied to a string and revolved in a vertical circle of radius r at a place where the acceleration due to gravity is g. At every instant of its motion, the body is acted upon by two forces, namely, its weight $\overrightarrow{mg}$ and the tension $\overrightarrow{T}$ in the string.

Let v_2 be the speed of the body and T_2 be the tension in the string at the lowest point B. We take the reference level for zero potential energy to be the bottom of the circle. Then, the body has only kinetic energy $\frac{1}{2}mv_2^2$ at the lowest point.

$\therefore T_2 = \frac{mv_2^2}{r} + mg$ ...(1)



Vertical circular motion (schematic)

and the total energy at the bottom = KE + PE

$$= \frac{1}{2}mv_2^2 + 0$$

$$= \frac{1}{2}mv_2^2 \quad ...(2)$$

Let v_1 be the speed and T_1 the tension in the string at the highest point A. As the body goes from B to A, it rises through a height h = 2r.

$$\therefore T_1 = \frac{mv_1^2}{r} - mg \quad .....(3)$$

and the total energy at A = KE + PE

$$= \frac{1}{2}mv_1^2 + mg(2r) \quad ...(4)$$

Then, from Eqs. (1) and (3),

$$T_2 - T_1 = \frac{mv_2^2}{r} + mg - \left(\frac{mv_1^2}{r} - mg\right)$$

$$= \frac{m}{r}\left(v_2^2 - v_1^2\right) + 2mg \quad ...(5)$$

Assuming that the total energy of the body is conserved, the total energy at the bottom = total energy at the top

Then, from Eqs. (2) and (4),

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 + mg(2r)$$

$$\therefore v_2^2 - v_1^2 = 4gr \quad ...(6)$$

Substituting this in Eq. (5),

$$T_2 - T_1 = \frac{m}{r}(4gr) + 2mg$$

$$= 4mg + 2mg$$

= 6 mg

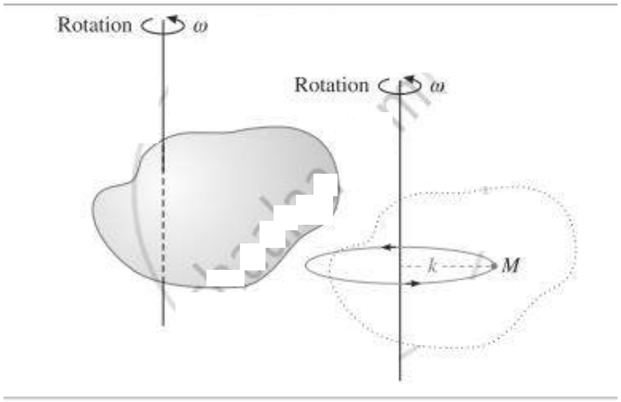
Therefore, the difference in the tensions in the string at the highest and the lowest points is 6 times the weight of the body.

Exercises | Q 5 | Page 24

Discuss the necessity of radius of gyration. Define it. On what factors does it depend and it does not depend? Can you locate come similarity between the center of mass and radius of gyration? What can you infer if a uniform ring and a uniform disc have the same radius of gyration?

SOLUTION

Definition: The radius of gyration of a body rotating about an axis is defined as the distance between the axis of rotation and the point at which the entire mass of the body can be supposed to be concentrated so as to give the same moment of inertia as that of the body about the given axis.



The moment of inertia (MI) of a body about a given rotation axis depends upon (i) the mass of the body and (ii) the distribution of mass about the axis of rotation. These two factors can be separated by expressing the MI as the product of the mass (M) and the square of a particular distance (k) from the axis of rotation. This distance is called the radius of gyration and is defined as given above. Thus,

$$I = \sum_i m_i r_i^2 = Mk^2$$

$$\therefore k = \sqrt{\frac{I}{M}}$$

Physical significance: The radius of gyration is less if I is less, i.e., if the mass is distributed close to the axis; and it is more if I is more, i.e., if the mass is distributed away from the axis. Thus, it gives an idea about the distribution of mass about the axis of rotation.

The centre of mass (CM) coordinates locates a point where if the entire mass M of a system of particles or that of a rigid body can be thought to be concentrated such that the acceleration of this point mass obeys Newton's second law of motion, viz.,

$\vec{F}_{net} = M\vec{a}_{CM}$, where $\vec{F}_{net}$ is the sum of all the external forces acting on the body or on the individual particles of the system of particles.

Similarly, radius of gyration locates a point from the axis of rotation where the entire mass M can be thought to be concentrated such that the angular acceleration of that point mass about the axis of rotation obeys the relation,

$\vec{\tau}_{net} = M\vec{\alpha}$, where $\vec{\tau}_{net}$ is the sum of all the external torques acting on the body or on the individual particles of the system of particles.

The radius of gyration of a thin ring of radius R_r about its transverse symmetry axis is

$$k_r = \sqrt{I_{CM}/M_r} = \sqrt{R_r^2} = R_r$$

The radius of gyration of a thin disc of radius R_d about its transverse symmetry axis is

$$k_d = \sqrt{I_{CM}/M_d} = \frac{\sqrt{M_d R_d^2/2}}{M_d} = \frac{1}{\sqrt{2}}R_d$$

Given $k_r = k_d$

$R_r = \frac{1}{R_d}$ or, equivalently, $R_d = \sqrt{2}R_r$.

Exercises | Q 6 | Page 24

Answer in brief:

State the conditions under which the theorems of parallel axes and perpendicular axes are applicable. State the respective mathematical expressions.

SOLUTION

The theorem of the parallel axis is applicable to anybody of arbitrary shape. The moment of inertia (MI) of the body about an axis through the center mass should be known, say, I_{CM}. Then, the theorem can be used to find the MI, I, of the body about an axis parallel to the above axis. If the distance between the two axes is h,

$I = I_{CM} + Mh^2$ ...(1)

The theorem of perpendicular axes is applicable to a plane lamina only. The moment of inertia Iz of a plane lamina about an axis – the z axis – perpendicular to its plane is equal to the sum of its moments of inertia I_x and I_x about two mutually perpendicular axes x and y in its plane and through the point of intersection of the perpendicular axis and the lamina.

$I_z = I_x + I_y$ ...(2)

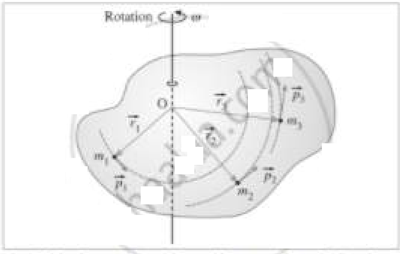
Exercises | Q 7 | Page 24

Answer in brief:

Derive an expression which relates angular momentum with the angular velocity of a rigid body.

SOLUTION

Consider a rigid body rotating with a constant angular velocity $\vec{\omega}$ about an axis through the point O and perpendicular to the plane of the figure. All the particles of the body perform uniform circular motion about the axis of rotation with the same angular velocity $\vec{\omega}$. Suppose that the body consists of N particles of masses m_1, m_2, ..., m_N situated at perpendicular distances r_1, r_2,..., r_N, respectively from the axis of rotation.



A rigid body rotating with a uniform angular velocity about an axis through O

The particle of mass m_1 revolves along a circle of radius r_1, with a linear velocity of magnitude $v_1 = r_1\omega$. The magnitude of the linear momentum of the particle is

$$p_1 = m_1v_1 = m_1r_1\omega$$

The angular momentum of the particle about the axis of rotation is by definition,

$$\vec{L}_1 = \vec{r}_1 \times \vec{p}_1$$

$$\therefore L_1 = r_1p_1\sin\theta$$

where θ is the smaller of the two angles between $\vec{r}_1$ and $\vec{p}_1$.

In this case, θ = 90°

∴ sin θ = 1

$$\therefore L_1 = r_1p_1 = r_1m_1r_1\omega = m_1r_1^2\omega$$

Similarly, $L_2 = m_2r_2^2\omega, L_3 = m_3r_3^2\omega$, etc.

The angular momentum of the body about the given axis is

$$L = L_1 + L_2 + ... + L_N$$

$$= m_1r_1^2\omega + m_2r_2^2\omega + ... + m_Nr_N^2\omega$$

$$= (m_1r_1^2 + m_2r_2^2 + ... + m_Nr_N^2)\omega$$

$$= \left(\sum_{i=1}^{N} m_i r_i^2 \right) \omega$$

∴ I = Iω

where $I = \sum_{i=1}^{N} m_i r_i^2$ = moment of inertia of the body about the given axis.

In vector form, $\vec{L} = I\vec{\omega}$

Thus, angular momentum = moment of inertia × angular velocity.

Exercises | Q 8 | Page 24

Obtain an expression for torque acting on a rotating body with constant angular acceleration. Hence state the dimensions and SI unit of torque.

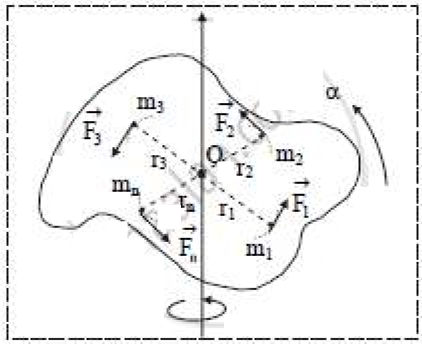
SOLUTION

a. Suppose a rigid body consists of n particles of masses m_1, m_2, m_3, ..., m_n which are situated at distances r_1, r_2, r_3, …, rn respectively, from the axis of rotation as shown in the figure.

b. Each particle revolves with angular acceleration α.

c. Let F_1, F_2, F_3, …., F_n be the tangential force acting on particles of masses, m_1, m_2, m_3, …, m_n respectively.

d. Linear acceleration of particles of masses m1, m2,…, mn are given by, $a_1 = r_1\alpha$, a2 = $r_2\alpha$, $a_3 = r_3\alpha$, …, a_n = rnα



e. Magnitude of force acting on particle of mass m1 is given by,

$\mathrm{F_1 = m_1 a_1 = m_1 r_1 \alpha} \qquad [\therefore \mathrm{a = r\alpha}]$

Magnitude of torque on particle of mass m1 is given by,

$\tau_1 = \mathrm{F_1 r_1 \sin\theta}$

But, $\theta = 90^\circ$ [$\because$ Radius vector is $\perp$ar to tangential force

$$\tau_1 = \mathrm{F_1 r_1 \sin 90^\circ}$$

$$= \mathrm{F_1 r_1}$$

$$= \mathrm{m_1 a_1 r_1}$$

$$\tau_1 = \mathrm{m_1 r_1^2 \alpha}$$

similarly

$$\tau_2 = \mathrm{m_2 r_2^2 \alpha}$$

$$\tau_3 = \mathrm{m_3 r_3^2 \alpha}$$

$$\tau_n = \mathrm{m_n r_n^2 \alpha}$$

Total torque acting on the body,

f. $\tau = \tau_1 + \tau_2 + \tau_3 + \ldots\ldots\ldots + \tau_n$

$$\tau = \mathrm{m_1 r_1^2 \alpha + m_2^2 \alpha + m_2 r_3^2 \alpha + \ldots\ldots + m_n r_n^2 \alpha}$$

$$\therefore \tau = \left[\sum_{i=1}^{n} \mathrm{m_i r_t^2}\right] \alpha$$

But $\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{m_i r_i^2 = I}$

$$\therefore \tau = I\alpha$$

g. Unit: Nm in SI system.

h. Dimensions: $\left[\mathrm{M^1 L^2 T^{-2}}\right]$

Exercises | Q 9 | Page 24

State and explain the principle of conservation of angular momentum. Use a suitable illustration. Do we use it in our daily life? When?

SOLUTION

Law (or principle) of conservation of angular momentum:

The angular momentum of a body is conserved if the resultant external torque on the body is zero.

Explanation: This law (or principle) is used by a figure skater or a ballerina to increase their speed of rotation for a spin by reducing the body's moment of inertia. A diver too uses it during a somersault for the same reason.

(1) Ice dance: Twizzle and spin are elements of the sport of figure skating. In a twizzle a skater turns several revolutions while travelling on the ice. In a dance spin, the skater rotates on the ice skate and centred on a single point on the ice. The torque due to friction between the ice skate and the ice is small. Consequently, the angular momentum of a figure skater remains nearly constant.

For a twizzle of a smaller radius, a figure skater draws her limbs close to her body to reduce the moment of inertia and increase the frequency of rotation. For larger rounds, she stretches out her limbs to increase the moment of inertia which reduces the angular and linear speeds.

A figure skater usually starts a dance spin in a crouch, rotating on one skate with the other leg and both arms extended. She rotates relatively slowly because her moment of inertia is large. She then slowly stands up, pulling the extended leg and arms to her body. As she does so, her moment of inertia about the axis of rotation decreases considerably, and thereby her angular velocity substantially increases to conserve angular momentum.

A figure skater performing a dance spin on the toes (Diagram for reference only)

(2) Diving: Take-off from a springboard or diving platform determines the diver's trajectory and the magnitude of angular momentum. A diver must generate angular momentum at take-off by moving the position of the arms and by a slight hollowing of the back. This allows the diver to change angular speeds for twists and somersaults in flight by controlling her/his moment of inertia. A compact tucked shape of the body lowers the moment of inertia for rotation of smaller radius and increased angular speed. The opening of the body for the vertical entry into water does not stop the rotation but merely slows it down. The angular momntum remains constant throughout the flight.

Exercises | Q 10 | Page 24

Discuss the interlink between translational, rotation and total kinetic energies of a rigid object rolls without slipping.

SOLUTION

Consider a symmetric rigid body, like a sphere or a wheel or a disc, rolling on a plane surface with friction along a straight path. Its center of mass (CM) moves in a straight line and, if the frictional force on the body is large enough, the body rolls without slipping. Thus, the rolling motion of the body can be treated as a translation of the CM and rotation about an axis through the CM. Hence, the kinetic energy of a rolling body is

$$E = E_{tran} + E_{rot} \quad ...(1)$$

where E_{tran} and E_{rot} are the kinetic energies associated with the translation of the CM and rotation about an axis through the CM, respectively.

Let M and R be the mass and radius of the body. Let ω, k and I be the angular speed, radius of gyration and moment of inertia for rotation about an axis through its centre, and v be the translational speed of the centre of mass.

$$\therefore v = \omega R \text{ and } I = Mk^2 \quad ...(2)$$

$$\therefore E_{tran} = \frac{1}{2}Mv^2 \text{ and } E_{rot} = \frac{1}{2}I\omega^2 \quad ...(3)$$

$$\therefore E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\frac{v^2}{R^2}$$

$$= \frac{1}{2}Mv^2\left(1 + \frac{I}{MR^2}\right) \quad ...(4)$$

$$= \frac{1}{2}Mv^2\left(1 + \frac{Mk^2}{MR^2}\right)$$

$$= \frac{1}{2}Mv^2\left(1 + \frac{k^2}{R^2}\right) \quad ..(5)$$

Exercises | Q 11 | Page 24

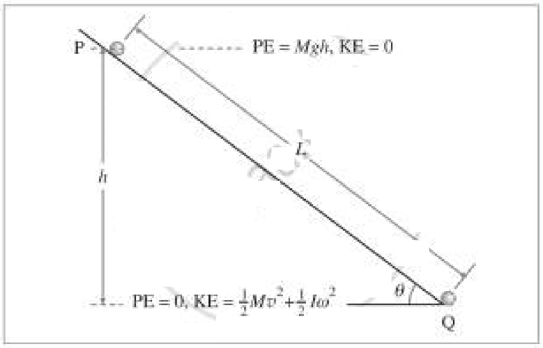
Answer in Brief:

A rigid object is rolling down an inclined plane derive the expression for the acceleration along the track and the speed after falling through a certain vertical distance.

SOLUTION

Consider a circularly symmetric rigid body, like a sphere or a wheel or a disc, rolling with friction down a plane inclined at an angle to the horizontal. If the frictional force on the body is large enough, the body rolls without slipping.

Let M and R be the mass and radius of the body. Let I be the moment of inertia of the body for rotation about an axis through its center. Let the body start from rest at the top of the incline at a



Rolling without slipping on an inclined plane

height h. Let v be the translational speed of the centre of mass at the bottom of the incline. Then, its kinetic energy at the bottom of the incline is

$$E = \frac{1}{2}MV^2\left[1 + \frac{1}{MR^2}\right] = \frac{1}{2}Mv^2(1+\beta) \quad ....(1)$$

where $\beta = \frac{I}{MR^2}$

If k is the radius of gyration of the body,

$I\ MK^2$ and $\beta = \frac{I}{MR^2} = \frac{k^2}{R^2}$

From the conservation of energy,

$$(KE + PE)_{initial} = (KE + PE)_{final} \quad ...(2)$$

$$\therefore 0 + Mgh = \frac{1}{2}Mv^2(1+\beta) + 0$$

$$\therefore Mgh = \frac{1}{2}Mv^2(1+\beta) \quad ...(3)$$

$$\therefore v^2 = \frac{2gh}{1+\beta}$$

$$\therefore v = \sqrt{\frac{2gh}{1+\beta}} = \sqrt{\frac{2gh}{1+\left(k^2/R^2\right)}} \quad .....(4)$$

Since h = L sin θ,

$$v = \sqrt{\frac{2gL\ \sin\theta}{1+\left(k^2/R^2\right)}} \quad ....(5)$$

Let a be the acceleration of the centre of mass of the body along the inclined plane. Since the body starts from rest,

$v_2 = 2aL$

$$\therefore a = \frac{v^2}{2L} \quad ....(6)$$

$$\therefore a = \frac{2gL\ \sin\theta}{1+\beta} . \frac{1}{2L} = \frac{g\ \sin\theta}{1+\beta} = \frac{g\ \sin\theta}{1+\left(k^2/R^2\right)} \quad ....(7)$$

Starting from rest, if t is the time taken to travel the distance L,

$$L = \frac{1}{2}at^2$$

$$\therefore t = \sqrt{\frac{2L}{a}} = \sqrt{\frac{2L}{g\ \sin\theta} . \left(1+\frac{k^2}{R^2}\right)} \quad .....(8)$$

[**Note:** For rolling without slipping, the contact point of the rigid body is instantaneously at rest relative to the surface of the inclined plane. Hence, the force of friction is static rather than kinetic, and does no work on the body. Thus, the force of static friction causes no decrease in the mechanical energy of the body and we can use the principle of conservation of energy.]

Exercises | Q 12 | Page 24

Somehow, an ant is stuck to the rim of a bicycle wheel of diameter 1 m. While the bicycle is on a central stand, the wheel is set into rotation and it attains the frequency of 2 rev/s in 10 seconds, with uniform angular acceleration. Calculate:

(i) The number of revolutions completed by the ant in these 10 seconds.
(ii) Time is taken by it for first complete revolution and the last complete revolution.

SOLUTION

Data: r = 0.5 m, $\omega_o = 0$, ω = 2rps, t= 10 s

(i) Angular acceleration (α) being constant, the average angular speed,

$$\omega_{av} = \frac{\omega_o + \omega}{2} = \frac{0+2}{2} = 1\text{rps}$$

∴ The angular displacement of the wheel in time t,

$\theta = \omega_{av} \cdot t = 1 \times 10 =$ **10 revolutions**

(ii) $\alpha = \frac{\omega - \omega_o}{t} = \frac{2-0}{10} = \frac{1}{5}\text{rev/s}^2$

$\theta = \omega_o t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \quad (\because \omega_o = 0)$

∴ For θ_1 = 1 rev,

$$1 = \frac{1}{2}\left(\frac{1}{5}\right)t\frac{2}{1}$$

$$\therefore t\frac{2}{1} = 10$$

$\therefore t_1 = \sqrt{10}\text{s} =$ **3.162s**

For θ_2 = 9 rev,

$$\therefore 9 = \frac{1}{2}\left(\frac{1}{5}\right)t\frac{2}{2}$$

$\therefore t_2 = \sqrt{90} = 3\sqrt{10} = 3(3.162) = 9.486$ s

The time for the last, i.e., the 10th, revolution is

$t - t_2 = 10 - 9.486 =$ **0.514 s**

Exercises | Q 13 | Page 24

The coefficient of static friction between a coin and a gramophone disc is 0.5. Radius of the disc is 8 cm. Initially the center of the coin is π cm away from the center of the disc. At what minimum frequency will it start slipping from there? By what factor will the answer change if the coin is almost at the rim?
(use $g = \pi^2 m/s^2$)

SOLUTION

Data: $\mu_s = 0.5$, $r_1 = \pi$ cm $= \pi \times 10^{-2}$ m, $r_2 = 8$ cm $= 8 \times 10^{-2}$ m, $g = \pi^2$ m/s^2

To revolve with the disc without slipping, the necessary centripetal force must be less than or equal to the limiting force of static friction.

$$\therefore m\omega^2 r \leq \mu s m g \therefore \omega^2 r \leq \mu s g$$

$$\therefore 4\pi^2 f_{min} r = \mu_s g \text{ ....}(\because \omega = 2\pi f) \text{ ....(1)}$$

$\therefore$ For $r = r_1$,

$$f^2_{min,1} = \frac{\mu_s g}{4\pi^2 r_1}$$

$$= \frac{(0.5)(\pi^2)}{4\pi^2(\pi \times 10^{-2})} = \frac{100}{8\pi} = \frac{25}{2\pi}$$

$$\therefore f^2_{min,1} = \sqrt{\frac{25}{2\pi}} = \frac{5}{\sqrt{2\pi}} \text{rps}$$

The coin will start slipping when the frequency is

$$\frac{5}{\sqrt{2\pi}} \text{rps}$$

From Eq. (1), $f^2_{min} \alpha \frac{1}{r}$

since μ_s and g are constant.

$$\therefore \frac{f_{min,2}}{f_{min,1}} = \sqrt{\frac{r_1}{r_2}} = \frac{\sqrt{\pi}}{8}$$

$$\therefore f_{min,2} = \sqrt{\frac{\pi}{8}} f_{min,1}$$

The minimum frequency in the second case will be $\sqrt{\frac{\pi}{8}}$ A times that in the first case.

[**Note:** The answers given in the textbook are for r_1 = 2cm.]

Exercises | Q 14 | Page 25

Answer in Brief:

Part of a racing track is to be designed for curvature of 72 m. We are not recommending the vehicles to drive faster than 216 kmph. With what angle should the road be tilted? By what height will its outer edge be, with respect to the inner edge if the track is 10 m wide?

SOLUTION

Given:

The radius of curvature of the track = 72 m

Maximum speed = 216 kmph = 60 m/s

Width of the track = 10 m

The angle of banking of the road is given by

$$\text{Tan } \theta = \frac{v^2}{rg}$$

$$\theta = \text{Tan}^{-1} \frac{v^2}{rg}$$

$$\theta = \text{Tan}^{-1} \left(\frac{60^2}{72 \times 10}\right)$$

$$\theta = \text{Tan}^{-1} \frac{3600}{720} = 5$$

$$\theta = \text{Tan}^{-1} 5 = 78.69°$$

The height of the outer edge of the road is given by

$\text{Sin } \theta = \frac{h}{w}$

$h = w \sin \theta$

$h = 10 \times \sin(78.69)$

$h = 10 \times 0.9805$

$h = 9.805 \text{ m}$

The angle that the road should be tilted is 78.69°.

The height of the outer edge is 9.805 m.

Exercises | Q 15 | Page 25

Answer in Brief:

The road in question 14 above is constructed as per the requirements. The coefficient of static friction between the tyres of a vehicle on this road is 0.8, will there be any lower speed limit? By how much can the upper-speed limit exceed in this case?

SOLUTION

Given:

$\mu_s = 0.8$

$r = 72 \text{ m}$

$\theta = 78°4'$

$g = 10 \text{ m/s}^2$

$\tan \theta = \tan 78°4' = 5$

$$V_{min} = \sqrt{rg\left[\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right]}$$

$$= \sqrt{72 \times 10\left(\frac{5 - 0.8}{1 + 0.8 \times 5}\right)}$$

$$= \sqrt{720 \times \frac{4.2}{4 + 1}}$$

$$= \sqrt{\frac{3024}{5}}$$

$$= \sqrt{144 \times 4.2}$$

= 12 × 2.049

= 24.588 m/s

= 88.52 km/h

This will be the lower limit or minimum speed on this track.

Since the track is heavily banked, $\theta > 45°$, there is no upper limit or maximum speed on this track.

Exercises | Q 16 | Page 25

Answer in Brief:

During a stunt, a cyclist (considered to be a particle) is undertaking horizontal circles inside a cylindrical well of radius 6.05 m. If the necessary friction coefficient is 0.5, how much minimum speed should the stunt artist maintain? The mass of the artist is 50 kg. If she/he increases the speed by 20%, how much will the force of friction be?

SOLUTION

Given data:

cylindrical well of radius r = 6.05 m

Coefficient of friction is μ = 0.5

Mass of the artist is m = 50 kg

What we have to find out:-

1) how much minimum speed should the stunt artist maintain V_{min} =?

2) How much will the force of friction be If she/he increases the speed by 20%, f =?

The minimum velocity to maintain motion is given by

$$v = \sqrt{\frac{rg}{\mu}}$$

$$V_{min} = \sqrt{\frac{6.05 \times 10}{0.5}} \quad .....[g = 10\ m/s^2]$$

$$= \sqrt{121}$$

$$\mathbf{V_{min} = 11\ m/s}$$

In well of death, the force of friction id depend on weight, you can refer from fig in textbook, hence we get,

$f = f_s = mg = 50 \times 10$

f = 500 N

Exercises | Q 17 | Page 25
Answer in Brief:
A pendulum consisting of a massless string of length 20 cm and a tiny bob of mass 100 g is set up as a conical pendulum. Its bob now performs 75 rpm. Calculate kinetic energy and increase in the gravitational potential energy of the bob. (Use $\pi^2 = 10$)

SOLUTION

Data: L = 0.2 m, m = 0.1 kg, $n = \frac{75}{60} = \frac{5}{4}$ rps,

$g = 10\ m/s^2,\ \pi^2 = 10,$

$T = \frac{1}{n} = \frac{4}{5}s = 0.8s$

$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

$\therefore T^2 = 4\pi^2\frac{L\cos\theta}{g}$

$\therefore h = L\cos\theta = \frac{gT^2}{4\pi^2} = \frac{(10)(0.8)^2}{4(10)} = 0.16\ m$ ...(1)

$\therefore \cos\theta = \frac{0.16}{0.2} = 0.8$

$\therefore \theta = \cos^{-1} 0.8 = 36.87° = 36°5'$

$v^2 = rg\tan\theta = (L\sin\theta)(g)\tan 36.87°$

$= (0.12)(10)(0.7500)$

$= 0.9$

The KE of the bob $= \frac{1}{2}\mathbf{mv}^2 = \frac{1}{2}(\mathbf{0.1})(\mathbf{0.9})$

= 0.045 J

The increase in gravitational PE,

Δ PE = mg (L - h)

= (0.1)(10)(0.2 - 0.16)

= 0.04 J

Exercises | Q 18 | Page 25

A motorcyclist (as a particle) is undergoing vertical circles inside a sphere of death. The speed of the motorcycle varies between 6 m/s and 10 m/s. Calculate the diameter of the sphere of death. How much minimum values are possible for these two speeds?

SOLUTION

Data: $\mathbf{v_{top}}$ = 6 m/s, $\mathbf{v_{bot}}$ = 10 m/s, g = 10 m/s^2

$$\mathbf{v_{bot}^2 = v_{top}^2 + 4gr}$$

$$\therefore r = \frac{\mathbf{v_{bot}^2 - v_{top}^2}}{\mathbf{4g}} = \frac{(\mathbf{10})^2 - (\mathbf{6})^2}{\mathbf{4 \times 10}} = \frac{\mathbf{64}}{\mathbf{40}}$$

= 1.6 m

The diameter of the sphere of death = 3.2 m.

For this r, $\mathbf{v_{min}} = \sqrt{\mathbf{gr}}$ at the top.

$\therefore \mathbf{v_{min}} = \sqrt{\mathbf{10 \times 1.6}} = \sqrt{\mathbf{16}}$ = 4 m/s

The corresponding minimum speed at the bottom

$= \sqrt{\mathbf{5gr}} = \sqrt{\mathbf{5(10)(1.6)}} = \sqrt{\mathbf{80}} = \mathbf{4\sqrt{5}}$ m/s

The required minimum values of the speeds are 4 m/s and $\mathbf{4\sqrt{5}}$ m/s.

Exercises | Q 19 | Page 25

A metallic ring of mass 1 kg has a moment of inertia 1 kg m^2 when rotating about one of its diameters. It is molten and remolded into a thin uniform disc of the same radius. How much will its moment of inertia be, when rotated about its own axis.

SOLUTION

mass of ring and disc is M =1 kg

Moment of inertia of ring at diameter (Ir)d = 1 kg m^2

Rr = Rd

What we have to find out:-

Moment of inertia of disc about own axis = Id =?

Using theorem of perpendicular axes, for a ring M.I about its axis passing through C.M and perpendicular to its plane is twice the M.I about its any diameter, which is given by,

(Ir)c = 2 (Ir)d

= 2 × 1

MRr^2 = 2 kg m^2

$Rr^2 = Rd^2$ = 2 meter

Hence,

Moment of inertia of disc about own axis is given by,

$$Id = \frac{1}{2} MRd^2$$

$$= \frac{1}{2} \times 1 \times 2$$

$$Id = 1 \text{ kg m}^2$$

Exercises | Q 20 | Page 25

Answer in Brief:

A big dumb-bell is prepared by using a uniform rod of mass 60 g and length 20 cm. Two identical solid spheres of mass 50 g and radius 10 cm each are at the two ends of the rod. Calculate the moment of inertia of the dumb-bell when rotated about an axis passing through its centre and perpendicular to the length.

SOLUTION

M_{sph} = 50 g, R_{sph} = 10 cm, M_{rod} = 60 g, L_{rod} = 20 cm

The MI of a solid sphere about its diameter is

$$I_{sph,CM} = \frac{2}{5}M_{sph}R_{sph}^2$$

The distance of the rotation axis (transverse symmetry axis of the dumbbell) from the centre of sphere, h = 30 cm.

The MI of a solid sphere about the rotation axis,

$$I_{sph} = I_{sph,CM} + M_{sph}h^2$$

For the rod, the rotation axis is its transverse symmetry axis through CM.

The MI of a rod about this axis,

$$I_{rod} = \frac{1}{12}M_{rod}L_{rod}^2$$

Since there are two solid spheres, the MI of the dumbbell about the rotation axis is

$i = 2I_{sph} + I_{rod}$

$$= 2M_{sph}\left(\frac{2}{5}R_{sph}^2 + h^2\right) + \frac{1}{12}M_{rod}L_{rod}^2$$

$$= 2(50)\left[\frac{2}{5}(10)^2 + (30)^2\right] + \frac{1}{12}(60)(20)^2$$

= 100 (40 + 900) + 5 (400)

= 94000 + 2000

= 96000 g cm^2

Exercises | Q 21 | Page 25

Answer in Brief:

A flywheel used to prepare earthenware pots is set into rotation at 100 rpm. It is in the form of a disc of mass 10 kg and a radius 0.4 m. A lump of clay (to be taken equivalent to a particle) of mass 1.6 kg falls on it and adheres to it at a certain distance x from the center. Calculate x if the wheel now rotates at 80 rpm.

SOLUTION

Data: f_1 = 100 rpm, f_2 = 80 rpm, M = 10 kg, R = 0.4 m, m = 1.6 kg

$I_1 = I_{wheel} = \frac{1}{2}MR^2 = \frac{1}{2}(10)(0.4)^2 = 0.8$ kg.m^2

The MI of the wheel and the lump of clay is

$I_2 = I_{wheel} + mx^2$

By the principle of conservation of angular momentum,

$I_1\omega_1 = I_2\omega_2$

$\therefore I_1(2\pi f_1) = I_2(2\pi f_2)$

$\therefore I_2 = I_{wheel} + mx^2 = \frac{f_1}{f_2}I_1 = \frac{f_1}{f_2}I_{wheel}$

$\therefore mx^2 = \left(\frac{f_1}{f_2} - 1\right)I_{wheel} = \left(\frac{100}{80} - 1\right)(0.8)$

$= \left(\frac{5}{4} - 1\right)(0.8) = 0.2$ kg.m^2

$v = x^2 = \frac{0.2}{1.6} = \frac{1}{8}$

$\therefore x = \frac{1}{\sqrt{8}}$ m = 0.3536 m

Exercises | Q 22 | Page 25

Starting from rest, an object rolls down along an incline that rises by 3 in every 5 (along with it). The object gains a speed of $\sqrt{10}$ m/s as it travels a distance of 5/3 m along the incline. What can be the possible shape/s of the object?

SOLUTION

Given data:

1) Incline that rises by 3 in every 5 is sin θ = $\frac{3}{5}$

2) Object gains a speed of v = $\sqrt{10}$ m/s

3) It travels a distance along the incline s = $\frac{5}{3}$ m

To find out:

Shape of possible object i.e to find out ratio of K^2/R^2 which will determine the possible rolling object.

We have,

Sin θ = $\frac{3}{5}$

And Linear distance travelled along the plane is s = $\frac{h}{\sin\theta}$

Hence,

h = s sin θ = $\frac{5}{3} \times \frac{3}{5} = 1$

The velocity of rolling body is given by,

v = $\sqrt{\frac{2gh}{1 + \frac{K^2}{R^2}}}$

Comparing we get,

$$\sqrt{10} = \sqrt{\frac{2gh}{1 + \frac{K^2}{R^2}}}$$

$$10 = \frac{2 \times 10 \times 1}{1 + \frac{K^2}{R^2}}$$

$$\left(1 + \frac{K^2}{R^2}\right) = 2$$

$$\frac{K^2}{R^2} = 1$$

Hence object must be Ring or hollow cylinder.